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Modified Josephson equations for the mesoscopic LC circuit including two coupled Josephson junctions*

Xiang-Guo Meng, Ji-Suo Wang and Bao-Long Liang

Department of Physics, Liaocheng University, Shandong 252059, People's Republic of China

E-mail: xgmeng@lcu.edu.cn

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Abstract

Based on the entangled state representation and Feynman's idea that 'electron pairs are bosons, . . . , a bound pair acts as a Bose particle', we present a cooper-pair number-phase quantization scheme for the mesoscopic LC circuit including two coupled Josephson junctions (JJs). Then we use the Heisenberg equation of motion to obtain the modified current equation and voltage equation across each JJ, as well as the equation for realizing quantum control. Besides, we investigate how the phases in two JJs are affected mutually through the capacitor coupling and the coupled JJs.

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1. Introduction

Recently, physicists delightedly found that a practical quantum computer might be built by using some solid-state devices, for instance, taking a Josephson junction (JJ) [1–4] as its core, which is comprised of two superconductors 'weakly' connected by a thin layer of insulating material [5, 6]. However, any practical quantum computer is composed of many mesoscopic circuits including, except for the JJs in series and in parallel, the capacitance, the inductance and electric resistance, etc. So many mesoscopic circuits including JJs are widely investigated [7–12]. The merits of the type of mesoscopic circuits lie in the following aspects: (1) they can exhibit strong entanglement owing to the entanglement of the JJ itself or the coupling between the JJ and the capacitor. Usually, the entanglement is recognized as an important physical resource in quantum information and many protocols are implemented on the basis of entangled state, so the circuits including JJs are widely applied in many information computation and processing. (2) The circuits with JJs can provide us with an artificial two-state system, which can be considered as a quantum qubit for designing a quantum computer.

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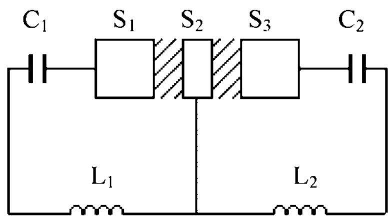

Figure 1. The mesoscopic LC circuit including two coupled Josephson junctions.

(3) Short-time decoherence of a single Josephson charge qubit [13] and many excellent properties of the single JJ in designs show the qualification for being a good candidate for quantum computation hardware.

In this paper, we propose the mesoscopic LC circuit including two coupled JJs (see figure 1). Supposing that at interval time $\Delta t \to 0$ the circuit is excited by an instant impluse source, then in a later process we shall investigate how the Josephson equations change because of the capacitor coupling and the coupling between two JJs. For this aim, following Feynman's explanation [5] about Cooper pair that 'a bound pair act as a Bose particle,$\cdots$', we present a bosonic operator Hamiltonian model of the mesoscopic LC circuit including two coupled JJs, then propose the Cooper-pair number-phase quantization scheme by virtue of the bipartite entangled state representation [14–16]. Furthermore, in the Heisenberg picture we obtain the modified Josephson current and voltage equations across the JJs, as well as the elementary realization of quantum control. The relations to the phase-difference operator $\hat{\varphi}_{j_1}$ and $\hat{\varphi}_{j_2}$ in the time evolution of the coupled JJs in series are also given.

2. Classical Hamiltonian analysis of the LC circuit including two coupled JJs

Let us begin with analyzing this circuit from the point of view of classical Hamiltonian dynamics. For the non-dissipative inductance, from the Faraday's law of electromagnetic induction we can obtain the voltage drop u_{L_l} across the lth inductance

$$u_{L_l} = \dot{\Phi}_{L_l}, \qquad l = 1, 2, \tag{1}$$

where Φ_{L_l} is the self-inductance magnetic flux through the lth inductance. For the lth single JJ, noticing that the current equation and the junction induction-voltage equation are

$$I_{j_l} = I_{c_l} \sin \varphi_{j_l}, \tag{2}$$

$$\dot{\varphi}_{j_l} = \frac{2eu_{j_l}}{\hbar}, \tag{3}$$

where $2e$ is the charge of a Cooper pair and E_{j_l} is the coupling energy of the lth junction, $I_{c_l} = 2eE_{j_l}/\hbar$ is the critical electric current of the JJ, the parameter u_{j_l} represents the voltage drop across the lth JJ and φ_{j_l} denotes the phase difference between two superconductors of the lth junction. So when the tunneling happens, the work done across the lth junction is

$$\int_0^t u_{j_l} I_{c_l} \sin \varphi_{j_l} \, \mathrm{d}t = E_{j_l} (\cos \varphi_{j_l} - 1). \tag{4}$$

Considering the direct and strong coupling between the latter two electrodes owing to the intermediate electrode is thin enough to allow the overlap of the superconducting order parameters of the outer layers, the coupling energy between superconductors 1 and 3 is $E_{j_3} = \hbar I_{13}/2e$, thus the whole system's potential energy is

$$\mathcal{V} = \sum_{l=1,2} \left[\frac{1}{2L_l} \Phi_{L_l}^2 + E_{j_l} \left(1 - \cos \varphi_{j_l}\right) \right] + E_{j_3} \left[1 - \cos \left(\varphi_{j_1} + \varphi_{j_2}\right)\right]. \tag{5}$$

On the other hand, taking C_{c_l} and C_{j_l} as the coupling capacitance and the junction capacitance respectively, the charging energy in all the capacitors is

$$\mathcal{T} = \sum_{l=1,2} \frac{1}{2} C_{c_l} u_{c_l}^2 + \frac{1}{2} C_{j_l} u_{j_l}^2, \tag{6}$$

where $\mathcal{T}$ is considered as the kinetic energy, since from equations (1) and (3) one can see that u_{j_l} and u_{c_l} are related to the generalized velocities $\dot{\varphi}_{j_l}$ and $\dot{\Phi}_{L_l}$, respectively. From the simple analysis of voltage at the nodes of the circuit we see

$$u_{c_l} = u_{j_l} + u_{L_l}. \tag{7}$$

Substituting equations (1), (3) and (7) into equation (6) yields

$$\mathcal{T} = \sum_{l=1,2} \frac{1}{2} \left(C_{c_l} + C_{j_l}\right) \left(\frac{\hbar}{2e}\right)^2 \dot{\varphi}_{j_l}^2 + \frac{1}{2} C_{c_l} \dot{\Phi}_{L_l}^2 + \frac{\hbar}{2e} C_{c_l} \dot{\varphi}_{j_l} \dot{\Phi}_{L_l}. \tag{8}$$

Then the Lagrangian function of the system is

$$\begin{aligned} \mathcal{L} &= \mathcal{T} - \mathcal{V} \\ &= \sum_{l=1,2} \left[\frac{1}{2} \left(C_{c_l} + C_{j_l}\right) \left(\frac{\hbar}{2e}\right)^2 \dot{\varphi}_{j_l}^2 + \frac{1}{2} C_{c_l} \dot{\Phi}_{L_l}^2 + \frac{\hbar}{2e} C_{c_l} \dot{\varphi}_{j_l} \dot{\Phi}_{L_l} \right. \\ &\qquad \left. - \frac{1}{2L_l} \Phi_{L_l}^2 - E_{j_l} \left(1 - \cos \varphi_{j_l}\right) \right] - E_{j_3} \left[1 - \cos \left(\varphi_{j_1} + \varphi_{j_2}\right)\right]. \end{aligned} \tag{9}$$

Owing to the charge neutrality law (or electric current continuity) at nodes of the circuit, $u_{j_l} C_{j_l} - 2n_l e = -u_{c_l} C_{c_l}$, then from equations (1), (3) and (7), we have

$$2n_l e = u_{j_l} C_{j_l} + u_{c_l} C_{c_l} = \left(C_{j_l} + C_{c_l}\right) \frac{\hbar}{2e} \dot{\varphi}_{j_l} + C_{c_l} \dot{\Phi}_{L_l}, \tag{10}$$

where n_l is the excess number of Cooper-pairs on the island. Using equations (1), (3), (9) and (10) we calculate the generalized momentum, respectively, conjugated to φ_{j_l} and Φ_{L_l}

$$p_{j_l} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_{j_l}} = \frac{\hbar}{2e} \left[\left(C_{j_l} + C_{c_l}\right) \frac{\hbar}{2e} \dot{\varphi}_{j_l} + C_{c_l} \dot{\Phi}_{L_l} \right] = n_l \hbar, \tag{11}$$

$$p_{L_l} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_{L_l}} = C_{c_l} \dot{\Phi}_{L_l} + \frac{\hbar}{2e} C_{c_l} \dot{\varphi}_{j_l}. \tag{12}$$

Note that the generalized momentum p_{j_l} is proportional to the excess number n_l of Cooper-pairs, which implies the possibility of number-phase quantization, as we will do in the next section. In terms of Legendre transformation and equations (11) and (12) we can obtain the classical Hamiltonian

$$H = \sum_{l=1,2} \left(p_{j_l} \dot{\varphi}_{j_l} + p_{L_l} \dot{\Phi}_{L_l}\right) - \mathcal{L} \equiv H_j + H_L + H_{\text{int}}, \tag{13}$$

where H_j is the Hamiltonian of two coupled JJs,

$$H_j = \sum_{l=1,2} \left[E_{c_l}^{(j)} n_l^2 + E_{j_l} \left(1 - \cos \varphi_{j_l}\right)\right] + E_{j_3} \left[1 - \cos \left(\varphi_{j_1} + \varphi_{j_2}\right)\right] \tag{14}$$

and the Hamiltonian H_L is equivalent to the superposition of two standard harmonic oscillators

$$H_L = \sum_{l=1,2} \frac{p_{L_l}^2}{2m_l} + \frac{1}{2L_l}\Phi_{L_l}^2, \tag{15}$$

as well as H_{int} is the coupling term

$$H_{\text{int}} = \sum_{l=1,2} \zeta_l n_l p_{L_l}, \tag{16}$$

where

$$E_{c_l}^{(j)} = \frac{2e^2}{C_{j_l}}, \qquad \frac{1}{m_l} = \frac{C_{j_l} + C_{c_l}}{C_{j_l} C_{c_l}}, \qquad \zeta_l = -\frac{2e}{C_{j_l}}. \tag{17}$$

3. The bosonic operator form of the classical Hamiltonian in (13)

Owing to the fact that the single LC circuit can be equivalent to a standard harmonic oscillator, as is mentioned above, we can obtain the quantum-mechanical Hamiltonian $\hat{H}_L$ by replacing p_{L_l}, Φ_{L_l} by the corresponding operators, with $\left[\hat{\Phi}_{L_l}, \hat{p}_{L_l}\right] = \mathrm{i}\hbar$. Thus the operator Hamiltonian $\hat{H}_L$ can be written in the standard bosonic form

$$\hat{H}_L = \sum_{l=1,2} \hbar\omega_l \left(\hat{c}_l^\dagger \hat{c}_l + 1/2\right), \tag{18}$$

where $\omega_l = \sqrt{\frac{C_{j_l}+C_{c_l}}{L_l C_{j_l} C_{c_l}}}$ is the characteristic frequency of the harmonic oscillator, and

$$\hat{c}_l^\dagger = \frac{1}{\sqrt{2m_l\hbar\omega_l}}\left(m_l\omega_l\hat{\Phi}_{L_l} - \mathrm{i}\hat{p}_{L_l}\right), \qquad \hat{c}_l = \frac{1}{\sqrt{2m_l\hbar\omega_l}}\left(m_l\omega_l\hat{\Phi}_{L_l} + \mathrm{i}\hat{p}_{L_l}\right) \tag{19}$$

are the bosonic operators, which satisfy the basic commutative relation $\left[\hat{c}_l, \hat{c}_l^\dagger\right] = 1$.

On the other hand, according to Feynman's view [5] that 'a bound pair acts as a Bose particle', we are naturally led to provide the bosonic operator model to quantize the classical Hamiltonian in (14). As we mentioned in the above section that the excess charge $2en_l$ should be quantized, so we let

$$n_l \to \hat{n}_l \equiv \hat{a}_l^\dagger \hat{a}_l - \hat{b}_l^\dagger \hat{b}_l, \tag{20}$$

be a Cooper-pair number operator in each box, $\hat{a}_l^\dagger$ and $\hat{b}_l^\dagger$ are Bose creation operators. To embody the number-phase quantization scheme we introduce the entangled state $|\eta\rangle_l$ [14–16]

$$|\eta\rangle_l = \exp\left[-\frac{1}{2}|\eta_l|^2 + \eta_l\hat{a}_l^\dagger - \eta_l^*\hat{b}_l^\dagger + \hat{a}_l^\dagger\hat{b}_l^\dagger\right]|00\rangle_l, \tag{21}$$

where $\eta_l = |\eta_l|\,\mathrm{e}^{\mathrm{i}\varphi_l}$, $|00\rangle_l$ is the two-mode vacuum state. The $|\eta\rangle_l$ state is constructed based on the idea of the quantum entanglement of Einstein, Podolsky and Rosen [17]. Using $\left[\hat{a}_l, \hat{a}_l^\dagger\right] = \left[\hat{b}_l, \hat{b}_l^\dagger\right] = 1$, we see that $|\eta\rangle_l$ obeys the eigenvector equations

$$\left(\hat{a}_l - \hat{b}_l^\dagger\right)|\eta\rangle_l = \eta_l|\eta\rangle_l, \qquad \left(\hat{b}_l - \hat{a}_l^\dagger\right)|\eta\rangle_l = -\eta^*|\eta\rangle_l. \tag{22}$$

The set of $|\eta\rangle_l$ makes up a complete quantum-mechanical representation

$$\int \frac{\mathrm{d}^2\eta_l}{\pi}|\eta\rangle_{ll}\langle\eta| = 1. \tag{23}$$

We can construct the bosonic phase operator for JJ [14]

$$\mathrm{e}^{\mathrm{i}\hat{\varphi}_l} = \sqrt{\frac{\hat{a}_l - \hat{b}_l^\dagger}{\hat{a}_l^\dagger - \hat{b}_l}}, \qquad \mathrm{e}^{-\mathrm{i}\hat{\varphi}_l} = \sqrt{\frac{\hat{a}_l^\dagger - \hat{b}_l}{\hat{a}_l - \hat{b}_l^\dagger}}, \qquad \cos\hat{\varphi}_l = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\hat{\varphi}_l} + \mathrm{e}^{-\mathrm{i}\hat{\varphi}_l}\right), \tag{24}$$

because in the $|\eta\rangle_l$ representation, $\mathrm{e}^{\mathrm{i}\hat{\varphi}_l}$ behaves as a phase

$$\mathrm{e}^{\mathrm{i}\hat{\varphi}_l}|\eta\rangle_l = \mathrm{e}^{\mathrm{i}\varphi_l}|\eta\rangle_l, \qquad \mathrm{e}^{-\mathrm{i}\hat{\varphi}_l}|\eta\rangle_l = \mathrm{e}^{-\mathrm{i}\varphi_l}|\eta\rangle_l. \tag{25}$$

Note $\left[\hat{a}_l^\dagger - \hat{b}_l, \hat{a}_l - \hat{b}_l^\dagger\right] = 0$, so they can reside in the same$\surd$. It then follows $\hat{\varphi}_l = \frac{1}{2\mathrm{i}}\ln\frac{\hat{a}_l - \hat{b}_l^\dagger}{\hat{a}_l^\dagger - \hat{b}_l}$, $\hat{\varphi}_l|\eta\rangle_l = \varphi_l|\eta\rangle_l$. From (22) we can derive

$$\begin{aligned}\hat{n}_l|\eta\rangle_l &\equiv \left(\hat{a}_l^\dagger\hat{a}_l - \hat{b}_l^\dagger\hat{b}_l\right)|\eta\rangle_l \\ &= \left[\hat{a}_l^\dagger\left(\eta_l + \hat{b}_l^\dagger\right) - \hat{b}_l^\dagger\left(a_l^\dagger - \eta_l^*\right)\right]|\eta\rangle_l = |\eta_l|\left(\hat{a}_l^\dagger\,\mathrm{e}^{\mathrm{i}\varphi_l} + \hat{b}_l^\dagger\,\mathrm{e}^{-\mathrm{i}\varphi_l}\right)|\eta\rangle_l \\ &= -\mathrm{i}\frac{\partial}{\partial\varphi_l}|\eta\rangle_l.\end{aligned} \tag{26}$$

so

$$[\hat{\varphi}_l, \hat{n}_l]|\eta\rangle_l = \left[\varphi_l, -\mathrm{i}\frac{\partial}{\partial\varphi_l}\right]|\eta\rangle_l = \mathrm{i}|\eta\rangle_l \to [\hat{\varphi}_l, \hat{n}_l] = \mathrm{i}, \tag{27}$$

which embodies number-phase quantization. Furthermore, using equations (25) and (26) we also derive the following commutative relations:

$$[\hat{n}_l, \cos\hat{\varphi}_l] = \mathrm{i}\sin\hat{\varphi}_l, \qquad [\hat{n}_l, \sin\hat{\varphi}_l] = -\mathrm{i}\cos\hat{\varphi}_l. \tag{28}$$

As a consequence of equations (25)–(27), the classical Hamiltonian in equation (14) is quantized as

$$\hat{H}_j = \sum_{l=1,2}\left[E_{c_l}^{(j)}\hat{n}_l^2 + E_{j_l}\left(1 - \cos\hat{\varphi}_{j_l}\right)\right] + E_{j_3}\left[1 - \cos\left(\hat{\varphi}_{j_1} + \hat{\varphi}_{j_2}\right)\right] \tag{29}$$

and the coupling term in equation (16) is quantized as

$$\hat{H}_{\mathrm{int}} = \sum_{l=1,2}\mathrm{i}\zeta_l\sqrt{m_l\hbar\omega_l/2}\,\hat{n}_l\left(\hat{c}_l^\dagger - \hat{c}_l\right). \tag{30}$$

4. Modified operator Josephson equations and quantum control

Using the Heisenberg equation of motion and the commutative relations (27) and (28) we can derive the equation of number operator in each JJ, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{n}_l = \frac{1}{\mathrm{i}\hbar}[\hat{n}_l, \hat{H}_j] = -\frac{E_{j_l}}{\hbar}\sin\hat{\varphi}_{j_l} - \frac{E_{j_3}}{\hbar}\sin\left(\hat{\varphi}_{j_1} + \hat{\varphi}_{j_2}\right), \qquad l = 1, 2 \tag{31}$$

thus equation (31) is equivalent to the Josephson current equation

$$-\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{Q}_l\rangle = \frac{2eE_{j_l}}{\hbar}\left\langle\sin\hat{\varphi}_{j_l}\right\rangle + \frac{2eE_{j_3}}{\hbar}\left\langle\sin\left(\hat{\varphi}_{j_1} + \hat{\varphi}_{j_2}\right)\right\rangle \equiv I_l, \qquad \hat{Q}_l = 2e\left(\hat{a}_l^\dagger\hat{a}_l - \hat{b}_l^\dagger\hat{b}_l\right), \tag{32}$$

which possess the coupling term and obviously differs from the current equation of the single JJ owing to the two JJs being coupled strongly. Similarly, the phase-difference operator $\hat{\varphi}_{j_l}$, conjugated to the Cooper-pair number-difference operator $\hat{n}_l$, evolves in the Heisenberg picture

$$\begin{aligned}\frac{\mathrm{d}\hat{\varphi}_{j_l}}{\mathrm{d}t} &= \frac{1}{\mathrm{i}\hbar}\left[\hat{\varphi}_{j_l}, \hat{H}_j + \hat{H}_{\mathrm{int}}\right] \\ &= \frac{1}{\mathrm{i}\hbar}E_{c_l}^{(j)}\left[\hat{\varphi}_{j_l}, \hat{n}_l^2\right] + \frac{1}{\mathrm{i}\hbar}\mathrm{i}\zeta_l\sqrt{m_l\hbar\omega_l/2}\left(\hat{c}_l^\dagger - \hat{c}_l\right)[\hat{\varphi}_{j_l}, \hat{n}_l] \\ &= \frac{1}{\hbar}\left(2E_{c_l}^{(j)}\hat{n}_l + \zeta_l\hat{p}_{L_l}\right),\end{aligned} \tag{33}$$

where we have used equations (19) and (27), and from which we see that the voltage equation is affected by the capacitor coupling as shown $\frac{1}{m_l} = \frac{C_{j_l}+C_{c_l}}{C_{j_l}C_{c_l}}$ in equation (17). Substituting equations (12) and (17) into equation (33) leads to

$$\frac{\mathrm{d}\hat{\varphi}_{j_l}}{\mathrm{d}t} = \frac{2e}{\hbar\left(C_{j_l}+C_{c_l}\right)}\left(2e\hat{n}_l - C_{c_l}\frac{\mathrm{d}\hat{\Phi}_{L_l}}{\mathrm{d}t}\right), \tag{34}$$

from which we see that the operator voltage equation about the junction is modified due to the capacitor coupling with the inductance. In fact this point can be further confirmed by deducing the Faraday operator equation from the Heisenberg equation

$$\hat{u}_{L_l} = \frac{\mathrm{d}\hat{\Phi}_{L_l}}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\left[\hat{\Phi}_{L_l}, \hat{H}_j + \hat{H}_{\mathrm{int}}\right] = \frac{1}{m_l}\hat{p}_{L_l} + \zeta_l\hat{n}_l. \tag{35}$$

Substituting equations (12) and (17) into equation (35) leads to

$$\hat{u}_{L_l} = \frac{\mathrm{d}\hat{\Phi}_{L_l}}{\mathrm{d}t} = \frac{2e}{C_{c_l}}\hat{n}_l - \frac{\hbar}{2e}\left(1+\frac{C_{j_l}}{C_{c_l}}\right)\frac{\mathrm{d}\hat{\varphi}_{j_l}}{\mathrm{d}t}. \tag{36}$$

This equation is the same as equation (34), which implies that $\frac{\mathrm{d}\hat{\Phi}_{L_l}}{\mathrm{d}t}$ and $\frac{\mathrm{d}\hat{\varphi}_{j_l}}{\mathrm{d}t}$ are nearly associated. Or we say that the Josephson equation is modified accompanying with the modification of Faraday equation about the inductance. Combining (31) and (34) we have

$$\frac{\mathrm{d}^2\hat{\varphi}_{j_l}}{\mathrm{d}t^2} = \frac{-2e}{\hbar\left(C_{j_l}+C_{c_l}\right)}\left\{\frac{2e}{\hbar}\left[E_{j_l}\sin\hat{\varphi}_{j_l} + E_{j_3}\sin\left(\hat{\varphi}_{j_1}+\hat{\varphi}_{j_2}\right)\right] + C_{c_l}\frac{\mathrm{d}^2\hat{\Phi}_{L_l}}{\mathrm{d}t^2}\right\}. \tag{37}$$

Owing to the contribution of the lth inductance is equivalent to the additional presence of a controllable gate voltage bias, equation (37) denotes that the variation of the voltage across the lth JJ is realized by controlling the gate voltage bias across the lth inductance and the gate capacitance C_{c_l}.

5. Relations between the phase difference $\hat{\varphi}_{j_1}$ and $\hat{\varphi}_{j_2}$

In the section we investigate the relations to the phase difference $\hat{\varphi}_{j_1}$ and $\hat{\varphi}_{j_2}$ involved in the time evolution of the single JJ when extra energy is applied to the junction. Supposing that an extra energy (say, light radiation) is applied to the first JJ, by comparing equation (33) with (3) we can see that the effective voltage drop across the first JJ is

$$\hat{U}_{j_1} = \frac{1}{2e}\left(2E_{c_1}^{(j)}\hat{n}_1 + \zeta_1\hat{p}_{L_1}\right). \tag{38}$$

Then in the interaction picture we can take the corresponding Hamiltonian as the following form,

$$\mathcal{H}_1' = \frac{1}{\hbar}\left(2E_{c_1}^{(j)}\hat{n}_1 + \zeta_1\hat{p}_{L_1}\right)\left[E_{j_1}\sin\hat{\varphi}_{j_l} + E_{j_3}\sin\left(\hat{\varphi}_{j_1}+\hat{\varphi}_{j_2}\right)\right], \tag{39}$$

which is the work done by the Josephson current through the first JJ in unit time interval in fact. According to the Heisenberg equation of motion in this picture, we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\sin\hat{\varphi}_{j_1} = \frac{1}{\mathrm{i}\hbar}\left[\sin\hat{\varphi}_{j_1}, \mathcal{H}_1'\right] = \frac{2E_{c_1}^{(j)}}{\hbar^2}\left[E_{j_1}\sin\hat{\varphi}_{j_1} + E_{j_3}\sin\left(\hat{\varphi}_{j_1}+\hat{\varphi}_{j_2}\right)\right]\cos\hat{\varphi}_{j_1} \tag{40}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\cos\hat{\varphi}_{j_1} = \frac{1}{\mathrm{i}\hbar}\left[\cos\hat{\varphi}_{j_1}, \mathcal{H}_1'\right] = -\frac{2E_{c_1}^{(j)}}{\hbar^2}\left[E_{j_1}\sin\hat{\varphi}_{j_1} + E_{j_3}\sin\left(\hat{\varphi}_{j_1}+\hat{\varphi}_{j_2}\right)\right]\sin\hat{\varphi}_{j_1}. \tag{41}$$

It then follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tan\frac{\hat{\varphi}_{j_1}}{2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1-\cos\hat{\varphi}_{j_1}}{\sin\hat{\varphi}_{j_1}}\right)$$
$$= -\lambda_1 E_{j_3}\sin\hat{\varphi}_{j_2}\tan^2\frac{\hat{\varphi}_{j_1}}{2} + 2\lambda_1\left(E_{j_1}+E_{j_3}\cos\hat{\varphi}_{j_2}\right)\tan\frac{\hat{\varphi}_{j_1}}{2} + \lambda_1 E_{j_3}\sin\hat{\varphi}_{j_2}, \tag{42}$$

where the parameter $\lambda_1 = \frac{E_{c_1}^{(j)}}{\hbar^2}$ and equation (42) shows that the variation of $\hat{\varphi}_{j_1}$ with the time t is affected by the phase $\hat{\varphi}_{j_2}$. The result is caused by the existence of the capacitor coupling and the coupling between two JJs. For the second JJ, in the interaction picture we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sin\hat{\varphi}_{j_2} = \frac{1}{\mathrm{i}\hbar}\left[\sin\hat{\varphi}_{j_2}, \mathcal{H}_1'\right] = 0, \tag{43}$$

which shows the phase difference $\hat{\varphi}_{j_2}$ does not change with the time t when an extra energy is only applied to the first JJ. So by a simple calculation we can derive the solutions of the equation (42) which obey the following relation,

$$2C\tan^2\frac{\hat{\varphi}_{j_1}}{2} + B\tan\frac{\hat{\varphi}_{j_1}}{2} + A\tanh\frac{A}{2}t - \left(2C\tan\frac{\hat{\varphi}_{j_1}(0)}{2} + B\right)\tan\frac{\hat{\varphi}_{j_1}(0)}{2} = 0, \tag{44}$$

where $\hat{\varphi}_{j_1}(0)$ is the initial phase value and the parameters are

$$A = 2\sqrt{E_{j_1}^2 + E_{j_3}^2 + 2E_{j_1}E_{j_3}\cos\hat{\varphi}_{j_2}}, \tag{45}$$

$$B = 2\left(E_{j_1} + E_{j_3}\cos\hat{\varphi}_{j_2}\right), \tag{46}$$

$$C = -E_{j_3}\sin\hat{\varphi}_{j_2}. \tag{47}$$

Similarly, when an extra energy is only applied to the second JJ,using equations (32) and (38) we can take the following Hamiltonian,

$$\mathcal{H}_2' = \frac{1}{\hbar}\left[2E_{c_2}^{(j)}\hat{n}_2 + \zeta_2\hat{p}_{L_2}\right]\left[E_{j_2}\sin\hat{\varphi}_{j_2} + E_{j_3}\sin\left(\hat{\varphi}_{j_1}+\hat{\varphi}_{j_2}\right)\right], \tag{48}$$

and we can also obtain the time evolvement of $\hat{\varphi}_{j_2}$, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\tan\frac{\hat{\varphi}_{j_2}}{2} = -\lambda_2 E_{j_3}\sin\hat{\varphi}_{j_1}\tan^2\frac{\hat{\varphi}_{j_2}}{2} + 2\lambda_2\left(E_{j_2}+E_{j_3}\cos\hat{\varphi}_{j_1}\right)\tan\frac{\hat{\varphi}_{j_2}}{2} + \lambda_2 E_{j_3}\sin\hat{\varphi}_{j_1}, \tag{49}$$

where $\lambda_2 = \frac{E_{c_2}^{(j)}}{\hbar^2}$ and equation (49) also shows how the phase $\hat{\varphi}_{j_2}$ in the second junction is affected by the phase $\hat{\varphi}_{j_1}$. Owing to the equation (49) about $\hat{\varphi}_{j_2}$ being similar to equation (42) in form and the phase $\hat{\varphi}_{j_1}$ is not affected by the extra energy applied to the second JJ, then its solution has the similar expression in equation (44).

On the other hand, due to the coupling capacitance C_{c_l}, the inductance and JJ should be considered in their totality. So for the inductance, though it is not directly radiated by light, we can still obtain the induction–voltage operator equation

$$\hat{u}_{L_l} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\Phi}_{L_l}$$
$$= \frac{1}{\mathrm{i}\hbar}\left[\Phi_{L_l}, \mathcal{H}_l'\right] = \frac{\zeta_l}{\hbar}\left[E_{j_l}\sin\hat{\varphi}_{j_l} + E_{j_3}\sin\left(\hat{\varphi}_{j_1}+\hat{\varphi}_{j_2}\right)\right], \qquad l = 1, 2 \tag{50}$$

which shows that the phase $\hat{\varphi}_{j_l}$ affects induction–voltage $\hat{u}_{L_l}$ across the lth inductance due to the existence of the coupling capacitor and the coupled JJs.

6. Conclusions

In summary, by virtue of the Hamilton dynamic approach we have obtained the classical Hamiltonian for the mesoscopic system composed of two LC circuits and two coupled JJs. The entangled state representation is used to propose Cooper-pair number-phase quantization and obtain the boson operator Hamiltonian for the whole system. Using the Heisenberg equation the operator Josephson equations are modified owing to the existence of the capacitor coupling and the coupled JJs, and we also show how to realize the variation of the voltage across the JJ by controlling the inductance and the capacitance. Besides, we have obtained the relations between the phase-difference $\hat{\varphi}_{j_1}$ and $\hat{\varphi}_{j_2}$ when one of the two JJs is affected by extra energy. Owing to the JJ being considered as a promising physical realization of solid state qubits, which are promising candidates for making processors of quantum computers, the above results may be useful in manufacturing quantum controlled devices and quantum computer as reference in theory.

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